

Uniform deviation and moment inequalities for random polytopes with general densities in arbitrary convex bodies

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Abstract

We prove an exponential deviation inequality for the convex hull of a finite sample of i.i.d. random points with a density supported on an arbitrary convex body in \mathbb{R}^d , $d \geq 2$. When the density is uniform, our result yields rate optimal upper bounds for all the moments of the missing volume of the convex hull, uniformly over all convex bodies of \mathbb{R}^d . We make no restrictions on their volume, location in the space or smoothness of their boundary. After extending an identity due to Efron, we also prove upper bounds for the moments of the number of vertices of the random polytope. Surprisingly, these bounds do not depend on the underlying density and we prove that the growth rates that we obtain are tight in a certain sense.

Keywords. convex body, convex hull, covering number, deviation inequality, random polytope, vertices

Mathematics Subject Classification 60D05, 52A22

1 Introduction

Probabilistic properties of random polytopes have been studied extensively in the literature in the last fifty years. Consider a collection of i.i.d. uniform random points in a convex body K in \mathbb{R}^d . Their convex hull is a random polytope whose volume and number of vertices have been first analyzed in the seminal work of Rényi and Sulanke [16, 17]. They derived the asymptotics of the expected volume in the case $d = 2$, when K is either a polygon,

with a given number of vertices, or a convex set with smooth boundary. More recently, considerable efforts were devoted to understanding the behavior of the expected volume. Several particular cases were investigated: For instance, when K is a d -dimensional simple polytope¹ [1], a d -dimensional polytope [2] or a d -dimensional Euclidean ball [6]. Groemer [9] (see also the references therein) proved that then K has volume one, the expected volume of the random polytope is minimum when K is an ellipsoid. Bárány and Larman [3] showed that if K has volume one, then one minus the expected volume has the same asymptotic behavior as the volume of the $(1/n)$ -wet part of K , defined as the union of all caps of K (a cap being the intersection of K with a half space) of volume at most $1/n$. Here, n is the number of uniform random points in K . This remarkable result reduces the initial probabilistic problem to computation of such a deterministic volume; This purely analytical problem was then extensively studied. When K has a smooth boundary, a key point was the introduction of the affine surface area, see [19, 20], and the volume of the $(1/n)$ -wet part is of the order $n^{-2/(d+1)}$. When K is a polytope, it is of a much smaller order, namely, $(\ln n)^{d-1}/n$ [3]; The expected volume is actually maximal when K is a simple polytope [3]. As a conclusion, the expectation of the volume is now very well-understood, when the underlying distribution is uniform. Much less is known about its higher moments and the tails of its distribution. Using a jackknife inequality for symmetric functions, Reitzner [15] proved that the boundary of K is smooth, the variance of the volume is bounded from above by $n^{-(d+3)/(d+1)}$, and he conjectured that this is the right order of magnitude for the variance. In addition, he proved that the second moment of the missing volume (i.e., the volume of K minus the volume of the random polytope) is exactly of the order $n^{-4/(d+1)}$, with explicit constants that depend on the affine surface area of K . Using martingales inequalities, Vu [22] obtained deviation inequalities for arbitrary convex bodies of volume one, involving quantities such as the volume of the wet part, and derived precise deviation inequalities in the two important cases when K is a polytope or has a smooth boundary. However, these inequalities involve constants which depend on K but are not explicit. As a consequence, upper bounds on the moments of the missing volume are proved, again with non explicit constants depending on K : Let K have volume one and V_n stand for the missing volume, then there exist

¹A d -dimensional simple polytope is a convex polytope such that each of its vertices is adjacent to exactly d edges.

positive constants α, c and ϵ_0 such that

$$\mathbb{P} \left[|V_n - \mathbb{E}[V_n]| \geq \sqrt{\lambda v} \right] \leq 2e^{-\lambda/4} + e^{-c\epsilon n}, \quad (1)$$

$$\forall \epsilon \in (\alpha(\ln n)/n, \epsilon_0], \lambda \in (0, n|K(\epsilon)|],$$

where $v = 36ng(\epsilon)^2|K(\epsilon)|$, $g(\epsilon) = \sup\{|F| : F \text{ star-shaped} \subseteq K(\epsilon)\}$, and $K(\epsilon)$ is the ϵ -wet part of K defined in [3]. Moreover, if K has a smooth boundary and volume one, there exist positive constants c and α , which depend on K , such that for any $\lambda \in \left(0, (\alpha/4)n^{-\frac{(d-1)(d+3)}{(d+1)(3d+5)}}\right]$, the following holds [22]:

$$\mathbb{P} \left[|V_n - \mathbb{E}[V_n]| \geq \sqrt{\alpha \lambda n^{-\frac{d+3}{d+1}}} \right] \leq 2 \exp(-\lambda/4) + \exp \left(-cn^{\frac{d-1}{3d+5}} \right). \quad (2)$$

This inequality yields upper bounds on the variance and on the q -th moment of the missing volume, respectively of orders $n^{-(d+3)/(d+1)}$ and $n^{-2q/(d+1)}$, for $q > 0$, for a *smooth convex body* K of volume one, up to constant factors that depend on K in an unknown way. Note that these two inequalities proved by Vu remain true when K has any positive volume, if $|V_n - \mathbb{E}[V_n]|$ is replaced by $|V_n - \mathbb{E}[V_n]|/|K|$. In our paper, we do not assume that the underlying distribution is uniform on K . We prove deviation inequalities and moment inequalities for a weighted missing volume, for general densities supported on the convex body K . In the uniform case, our results yield a deviation inequality which, unlike (1) and 2, which hold for a very small range of λ , captures the whole tail of the distribution of V_n . Our inequality is uniform over all convex bodies K , no matter their volume and boundary structure, and our constants do not depend on K . Our approach is based on a very simple covering number argument and is not bonded to the uniform distribution, which, to our best knowledge, makes our deviation inequalities completely new.

In addition, we derive moment inequalities for the number of vertices of the random polytope. In the uniform case, we prove that the rates in our upper bounds are tight, uniformly on all convex bodies. As a consequence, we also prove that the growth of the moments of the number of vertices is the highest when the underlying density is uniform.

2 Notation and statement of the problem

Let $d \geq 2$ be an integer. We denote by $|\cdot|$ the Lebesgue measure in \mathbb{R}^d , ρ the Euclidean distance in \mathbb{R}^d , B_d the unit Euclidean ball with center 0, and

β_d its volume.

If $G \subseteq \mathbb{R}^d$ and $\epsilon > 0$, we denote by $G^\epsilon = \{x \in \mathbb{R}^d : \rho(x, G) \leq \epsilon\}$ the closed ϵ -neighborhood of G . Here, $\rho(x, G) = \inf_{y \in G} \rho(x, y)$. If G is measurable, we denote by $|G|$ its volume.

The symmetric difference between two sets G_1 and G_2 is denoted by $G_1 \triangle G_2$ and their Hausdorff distance is denoted and defined as:

$$d_H(G_1, G_2) = \inf\{\epsilon > 0 : G_1 \subseteq G_2^\epsilon, G_2 \subseteq G_1^\epsilon\}.$$

We denote by \mathcal{K}_d the class of all convex bodies in \mathbb{R}^d , and by \mathcal{K}_d^1 the collection of all those included in B_d . The convex hull of n i.i.d. random points X_1, \dots, X_n is denoted by \hat{K}_n . If X_1, \dots, X_n have a density f with respect to the Lebesgue measure in \mathbb{R}^d , we denote by \mathbb{P}_f their joint probability measure and by \mathbb{E}_f the corresponding expectation operator (we omit the dependency in n unless stated otherwise). If f is the uniform density on a convex body K , we rather use the notation \mathbb{P}_K and \mathbb{E}_K . In general, when the density f of X_1 is supported on a convex body K , we denote by $d_f(K, \hat{K}_n) = \int_{K \setminus \hat{K}_n} f(x) dx$ and by V_n the missing volume of \hat{K}_n , i.e., $V_n = |K| - |\hat{K}_n|$. The integral $d_f(K, \hat{K}_n)$ can be interpreted as a weighted missing volume. We are interested in deviation inequalities for Z , where Z is either $d_f(K, \hat{K}_n)$ or V_n , i.e., in bounding from above $\mathbb{P}_K[Z > \epsilon]$, for $\epsilon > 0$. We are also interested in upper bounds for the moments $\mathbb{E}_K[Z^q]$, $q > 0$. Our main result is stated in Section 3: We prove a deviation inequality for the weighted missing volume, and we investigate a special class of densities, satisfying the so called *margin condition*, for which we are also able to control the unweighted missing volume. In Section 4, we investigate the moments of the number of vertices of the random polytope, with no restrictions on K and on the underlying density on K , as long as it is bounded from above. Finally, in Section 5, we focus on the uniform case, and we derive a deviation inequality for the missing volume, and prove that the rates of the subsequent moment inequalities are tight. Last section is devoted to some proofs.

3 Deviation inequality for the missing volume of random polytopes

Our main result is the following theorem.

Theorem 1. *Let $n \geq 1$. Let $K \in \mathcal{K}_d^1$, f be a density supported on K and X_1, \dots, X_n be i.i.d. random points with density f . Assume that $f \leq M$ almost everywhere, for some positive number M . Then, there exist positive constants C_1 and C_2 that depend on d only, such that the following holds.*

$$\mathbb{P} \left[n(d_f(K, \hat{K}_n) - C_1(M+1)n^{-2/(d+1)}) > x \right] \leq C_2 e^{-x}, \quad \forall x \geq 0.$$

Proof. This proof is inspired by Theorem 1 in [11], which derives an upper bound for the risk of a convex hull type estimator of a convex function. It is based on an upper bound of the covering number of \mathcal{K}_d^1 , proven by [4]. For $\delta > 0$, a δ -net of \mathcal{K}_d^1 for the Hausdorff distance is a finite subset \mathcal{N}_δ of \mathcal{K}_d^1 such that for all $G \in \mathcal{K}_d^1$, there exists $G^* \in \mathcal{N}_\delta$ with $d_H(G, G^*) \leq \delta$. The covering number of \mathcal{K}_d^1 for the Hausdorff distance is the function that maps $\delta > 0$ to the minimum cardinality of a δ -net of \mathcal{K}_d^1 for the Hausdorff distance. The following lemma is proven in [4].

Lemma 1. *The covering number of \mathcal{K}_d^1 for the Hausdorff distance is not larger than $c_1 e^{\delta^{-(d-1)/2}}$, for all $\delta > 0$, where $c_1 > 0$ depends on d only.*

Our next lemma shows that the Nykodim distance (i.e., the volume of the symmetric difference) between two sets in \mathcal{K}_d^1 is dominated by their Hausdorff distance. The proof is deferred to the appendix.

Lemma 2. *There exists a positive constant α_1 which depends on d only, such that*

$$|G \triangle G'| \leq \alpha_1 d_H(G, G'), \quad \forall G, G' \in \mathcal{K}_d^1.$$

Let $\delta = n^{-2/(d+1)}$ and $\{K_1, \dots, K_N\}$ be a δ -net of \mathcal{K}_d^1 , where N is a positive integer satisfying $N \leq c_1 e^{\delta^{-(d-1)/2}}$, cf. Lemma 1. Let $\hat{j} \leq N$ be such that $d_H(\hat{K}_n, K_{\hat{j}}) \leq \delta$. By Lemma 2, this implies that $|\hat{K}_n \setminus K_{\hat{j}}| \leq |\hat{K}_n \triangle K_{\hat{j}}| \leq \alpha_1 \delta$ and hence, since f is nonnegative,

$$d_f(K, \hat{K}_n) \leq \int_{K \setminus K_{\hat{j}}} f + \int_{K_{\hat{j}} \setminus \hat{K}_n} f \leq \int_{K \setminus K_{\hat{j}}} f + \alpha_1 M \delta.$$

In addition, since $d_H(\hat{K}_n, K_{\hat{j}}) \leq \delta$, it is true that $\hat{K}_n \subseteq K_{\hat{j}}^\delta$, yielding $X_i \in$

K_j^δ , for all $i = 1, \dots, n$. Therefore, for all $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\mathbb{P}_f \left[d_f(K, \hat{K}_n) > \varepsilon \right] &\leq \mathbb{P}_f \left[d_f(K, K_{\hat{j}}) > \varepsilon - \alpha_1 M \delta \right] \\
&\leq \mathbb{P}_f \left[\exists j = 1, \dots, N : d_f(K, K_j) > \varepsilon - M \delta \right. \\
&\quad \left. \text{and } X_i \in K_j^\delta, \forall i = 1, \dots, n \right] \\
&\leq \sum_{j \in I_{\varepsilon - \alpha_1 M \delta}} \mathbb{P}_f \left[X_1 \in K_j^\delta \right]^n \\
&= \sum_{j \in I_{\varepsilon - \alpha_1 M \delta}} \left(\int_{K_j^\delta} f \right)^n, \tag{3}
\end{aligned}$$

where we used the union bound, and for $\eta \in \mathbb{R}$, we denoted by $I_\eta = \{j = 1, \dots, N : d_f(K, K_j) > \eta\}$. Note that

$$\int_{K_j^\delta} f = \int_{K_j} f + \int_{K_j^\delta \setminus K_j} f \leq 1 - d_f(K, K_j) + |K_j^\delta \setminus K_j| M. \tag{4}$$

By Lemma 2, the last term is bounded from above by $\alpha_1 \delta$ and (4) entails, if $j \in I_{\varepsilon - \alpha_1 M \delta}$,

$$\int_{K_j^\delta} f \leq 1 - \varepsilon + 2\alpha_1 M \delta.$$

Hence, (3) becomes

$$\begin{aligned}
\mathbb{P}_f \left[d_f(K, \hat{K}_n) > \varepsilon \right] &\leq N(1 - \varepsilon + 2\alpha_1 M \delta)^n \\
&\leq c_1 \exp \left(-n(\varepsilon - 2\alpha_1 M \delta) + \delta^{-(d-1)/2} \right) \\
&= c_1 \exp \left(-n(\varepsilon - (2\alpha_1 M + 1)\delta) \right). \tag{5}
\end{aligned}$$

Note that since $d_f(K, \hat{K}_n) \leq 1$ almost surely, (5) actually holds for all $\varepsilon > 0$ (we have assumed $\varepsilon \in (0, 1)$ so far). This ends the proof by taking ε of the form $\frac{x}{n} + (2\alpha_1 M + 1)\delta$. □

As a consequence of Theorem 1, we get upper bounds for all the moments of $d_f(K, \hat{K}_n)$.

Corollary 1. *Let the assumptions of Theorem 1 hold. Then, for all $q > 0$, there exists $A_q > 0$ that depends on q and d only such that*

$$\mathbb{E}_f \left[d_f(K, \hat{K}_n)^q \right] \leq A_q (M + 1)^q n^{-2q/(d+1)}.$$

Proof. The proof is based on an application of Fubini's theorem. Namely, if Z is a nonnegative random variable and $q > 0$, then

$$\mathbb{E}[Z^q] = q \int_0^\infty t^{q-1} \mathbb{P}[Z > t] dt.$$

Let $Z = d_H(K, \hat{K}_n)$ and denote by $\delta = C_1(M+1)n^{-2/(d+1)}$. Then,

$$\begin{aligned} \mathbb{E}_f[Z^q] &= q \int_0^\infty t^{q-1} \mathbb{P}[Z > t] dt \\ &= q \int_0^\delta t^{q-1} \mathbb{P}[Z > t] dt + q \int_\delta^\infty t^{q-1} \mathbb{P}[Z > t] dt \\ &\leq \delta^q + q \int_0^\infty (t + \delta)^{q-1} \mathbb{P}[Z > t + \delta] dt \\ &= \delta^q + \frac{q}{n} \int_0^\infty \left(\frac{x}{n} + \delta\right)^{q-1} \mathbb{P}[Z > x/n + \delta] dx \\ &\leq \delta^q + \frac{C_2 q}{n} \int_0^\infty \left(\frac{x}{n} + \delta\right)^{q-1} e^{-x} dx \\ &\leq \delta^q + \frac{C_2 q}{n} \int_0^\infty 2^{q-2} \left(\frac{x^{q-1}}{n^{q-1}} + \delta^{q-1}\right) e^{-x} dx \\ &\leq a_q \delta^q, \end{aligned}$$

for some constant a_q that depends on d and q only. □

The inequalities that we have obtained for $d_f(K, \hat{K}_n)$ can transfer to the missing volume $|K \setminus \hat{K}_n|$ under some conditions on f . An important such condition, called the *margin condition* (see [14, 21]), is the following. For $t > 0$, let $K_f(t) = \{x \in K : f(x) \leq t\}$. The density f satisfies the margin condition with parameters $\alpha \in (0, \infty]$, $L, t_0 > 0$ if and only if

$$|K_f(t)| \leq Lt^\alpha,$$

for all $t \in (0, t_0]$. The case $\alpha = \infty$ corresponds to a density f that is almost everywhere bounded away from zero on K . Let us give two other important cases where a margin condition is satisfied.

Slow decay of f near the boundary of K : Assume that f does not decay too fast near the boundary of K (which we denote by ∂K). Namely, assume the existence of positive numbers ρ_0, c and γ such that for all $x \in K$,

$$f(x) \geq c \min(\rho_0, \rho(x, \partial K))^\gamma. \quad (6)$$

Then, f satisfies the margin condition with $t_0 = c\rho_0^\gamma$, $L = \frac{\kappa}{c^{1/\gamma}}$ and $\alpha = 1/\gamma$, where κ is any number that no smaller than the surface area of K (e.g., take κ to be the surface area of the unit ball, if $K \in \mathcal{K}_d^1$).

Projection of higher dimensional convex bodies: Let $D > d$ be an integer and $K_0 \in \mathcal{K}_D^1$. Let Y_1, \dots, Y_n be i.i.d. uniform random points in K_0 . Identify \mathbb{R}^d with a linear subspace of \mathbb{R}^D and let H be its orthogonal space in \mathbb{R}^D . Let K be the orthogonal projection of K_0 onto \mathbb{R}^d and let X_i be the orthogonal projection of Y_i onto \mathbb{R}^d , for $i = 1, \dots, n$. Assume that K_0 satisfies the r -rolling ball condition, where $r > 0$: Namely, assume that for all $x \in \partial K_0$, there exists $a \in K_0$ with $x \in B_D(a, r) \subseteq K_0$. Then, we have the following lemma, whose proof is deferred to the appendix.

Lemma 3. *The density f of the X_i 's satisfies (6), with $\rho_0 = r$, $c = r^{(D-d)/2}\beta_D\beta_{D-d}$ and $\gamma = (D-d)/2$.*

Hence, as we already saw in the previous example, f satisfies the margin condition with $\alpha = 2/(D-d)$.

The following lemma gives a (deterministic) control of $d_f(K, \hat{K}_n)$ on the missing volume. For completeness of the presentation, we provide its proof in the appendix (see also Proposition 1 in [21]).

Lemma 4. *Let f satisfy the margin condition with parameters α, L, t_0 . Assume that $d_f(K, \hat{K}_n) \leq t_0^{\alpha+1}$. Then,*

$$|K \setminus \hat{K}_n| \leq (L+1)d_f(K, \hat{K}_n)^{\alpha/(\alpha+1)}.$$

If f satisfies a margin condition, the deviation and moment inequalities that we have for $d_f(K, \hat{K}_n)$ transfer to the missing volume, as shown in the next two results.

Theorem 2. *Let f satisfy the margin condition with parameters α, L and t_0 and assume that $f \leq M$ for some positive number M . Then, there exists a positive integer n_0 that depends on d, t_0 and M and positive constants C_3 and C_4 that depend on d, α and L such that, for all $n \geq n_0$ and for all $x \geq 0$,*

$$\mathbb{P}_f \left[n^{\frac{\alpha}{\alpha+1}} \left(|K \setminus \hat{K}_n| - C_3 n^{-\frac{2\alpha}{(\alpha+1)(d+1)}} \right) > x \right] \leq C_4 e^{-x^{(\alpha+1)/\alpha}} + C_4 e^{-nt_0/2}.$$

Note that the constant C_4 is the same as C_2 in Theorem 1 and that n_0 is the first integer n that satisfies $C_1(M+1)n^{-2/(d+1)} \leq t_0/2$, where C_1 is defined in Theorem 1.

Proof. For $\varepsilon > 0$, write

$$\begin{aligned} & \mathbb{P}_f \left[|K \setminus \hat{K}_n| > \varepsilon \right] \\ &= \mathbb{P}_f \left[|K \setminus \hat{K}_n| > \varepsilon, d_f(K, \hat{K}_n) \leq t_0 \right] + \mathbb{P}_f \left[|K \setminus \hat{K}_n| > \varepsilon, d_f(K, \hat{K}_n) > t_0 \right] \\ &\leq \mathbb{P}_f \left[d_f(K, \hat{K}_n) > \varepsilon^{(\alpha+1)/\alpha} \right] + \mathbb{P}_f \left[d_f(K, \hat{K}_n) > t_0 \right] \end{aligned} \quad (7)$$

and apply Theorem 1 to get the desired result. \square

As a consequence of the deviation inequality of Theorem 2, we get the following moment inequalities.

Corollary 2. *Recall the notation and assumptions of Theorem 2. Then, for all $q > 0$, there exists a positive constant A'_q that depends on d, α, L, t_0 and q only, such that*

$$\mathbb{E}_f \left[|K \setminus \hat{K}_n|^q \right] \leq A'_q (M+1)^q n^{-\frac{2\alpha q}{(\alpha+1)(d+1)}}, \quad \forall n \geq n_0.$$

Proof. The proof is based on the same argument as in the proof of Corollary 1 and is omitted. \square

It is easy to see that the constants C_3 and C_4 in Theorem 2 are bounded, as functions of α . Hence, when $\alpha = \infty$, which includes the case of the uniform distribution on K , the rate obtained in Theorem 2 coincides with that obtained in Section 5. As a byproduct, the deviation inequality given in Theorem 4 below, for the missing volume, still holds, with different constants, for any density that is bounded away from zero and infinity (see Remark 2 in Section 5).

4 Moment inequalities for the number of vertices of random polytopes

In this section, we are interested in the number of vertices of random polytopes. Let μ be any probability measure in \mathbb{R}^d . Let X_1, X_2, \dots be i.i.d. realizations of μ and \hat{K}_n be the convex hull of the first n of them, for $n \geq 1$. Denote by \mathcal{V}_n the set of vertices of \hat{K}_n and R_n its cardinality, i.e., the number of vertices of \hat{K}_n . Efron [7] proved a simple but elegant identity, which relates the expected missing mass $\mathbb{E} \left[1 - \mu(\hat{K}_n) \right]$ to the expected number of vertices $\mathbb{E}[R_{n+1}]$ of \hat{K}_{n+1} . Namely, one has

$$\mathbb{E} \left[1 - \mu(\hat{K}_n) \right] = \frac{\mathbb{E}[R_{n+1}]}{n+1}, \quad \forall n \in \mathbb{N}^*. \quad (8)$$

In the case when μ is the uniform probability measure on a convex body K , extensions of this identity to higher moments of $|K \setminus \hat{K}_n|$ can be found in [5]. Here, we prove the following inequalities that hold for any distribution μ .

Lemma 5. *For all positive integer q ,*

$$\mathbb{E} \left[\prod_{j=0}^{q-1} \frac{R_{n+q} - j}{n + q - j} \right] \leq \mathbb{E} \left[\left(1 - \mu(\hat{K}_n) \right)^q \right].$$

If μ has a bounded density f with respect to the Lebesgue measure and is supported on a convex body K , we can combining Corollary 3 and Theorem 5 yields the following inequality:

$$\mathbb{E}_f [R_n(R_n - 1) \dots (R_n - q + 1)] \leq A_q(M + 1)^q n^{\frac{q(d-1)}{d+1}}, \forall n \in \mathbb{N}^*, \forall q \in \mathbb{N}^*,$$

where A_q is the same constant as in Corollary 3 and M is an almost everywhere upper bound of f . Since the polynomial x^q is a linear combination of the polynomials $x(x-1)\dots(x-k+1)$, $0 \leq k \leq q$, we get the following result.

Theorem 3. *Let K be a convex body and f satisfy the assumptions of Theorem 1. Then, for all positive integer q , there exists a positive constant B_q that depends on d and q only such that*

$$\mathbb{E}_f [R_n^q] \leq B_q(M + 1)^q n^{\frac{q(d-1)}{d+1}}. \quad (9)$$

Remark 1. *The boundedness assumption on f in Theorem 3 seems unavoidable, in the following sense: Let μ be a probability measure that puts positive mass on arbitrarily many points in K , near its boundary and let f be the density of a regularized version of μ , truncated so it remains supported on K . Then, with high probability, R_n can be arbitrarily large.*

5 The case of uniform distributions

As discussed in the introduction, the case of the uniform distribution on a convex body K is an extremely important case in the stochastic geometry literature. In this section, we use the results proven in the previous sections in order to derive universal inequalities for the convex hull of uniform random points. By universal, we mean uniform for all convex bodies, irrespective of their volume, or facial structure.

Theorem 4. *There exist two positive constants \aleph_1, \aleph_2 and \aleph_3 , which depend on d only, such that:*

$$\sup_{K \in \mathcal{K}_d} \mathbb{P}_K \left[n \left(\frac{|K \setminus \hat{K}_n|}{|K|} - \aleph_1 n^{-2/(d+1)} \right) > x \right] \leq \aleph_2 e^{-\aleph_3 x}, \forall x > 0. \quad (10)$$

Proof. In order to prove Theorem 4, we first state two lemmas, the first of which is about the so called *John's ellipsoid* of a convex body.

Lemma 6. *For all $K \in \mathcal{K}_d$, there exists $a \in \mathbb{R}^d$ and an ellipsoid E such that*

$$a + d^{-1}(E - c) \subseteq K \subseteq E. \quad (11)$$

Proof of Lemma 6 can be found in [12] and [10]. If E is an ellipsoid of maximum volume satisfying (11) for some $a \in \mathbb{R}^d$, then $a + d^{-1}(E - c)$ is called *John ellipsoid* of K .

Let $K \in \mathcal{K}_d$ and X_1, \dots, X_n be i.i.d. uniform random points in K . Let E be an ellipsoid that satisfies (11), and T an affine transform in \mathbb{R}^d which maps E to the unit ball B_d . Note that TX_1, \dots, TX_n are independent and uniformly distributed in TK , and their convex hull is $T\hat{K}_n$. Hence, the distribution of $\frac{|K \setminus \hat{K}_n|}{|K|} = \frac{|TK \setminus T\hat{K}_n|}{|TK|}$ is the same as that of $\frac{|TK \setminus \hat{K}'_n|}{|TK|}$, where \hat{K}'_n is the convex hull of X'_1, \dots, X'_n , which are i.i.d. uniform random points in TK . Therefore, for all $\varepsilon > 0$,

$$\mathbb{P}_K \left[\frac{|K \setminus \hat{K}_n|}{|K|} > \varepsilon \right] = \mathbb{P}_f[d_f(TK, \hat{K}_n) > \varepsilon], \quad (12)$$

where f is the uniform density on TK . The density f is bounded from above by $M = 1/|TK| \leq d^d/|TE| = d^d/\beta_d$, by definition of E and T . Hence, applying Theorem 1 yields the desired result, since all the constants in that theorem depends on d only. \square

Remark 2. *Similar arguments could be used to prove a deviation inequality with the same rate $n^{-2/(d+1)}$, for a density f that is nearly uniform on K , i.e., that satisfies $0 < m \leq f(x) \leq M$ for all $x \in K$, where m and M are positive numbers. Indeed, for all invertible affine transformation T , $d_g(TK, \hat{K}'_n)$ and $d_f(K, \hat{K}_n)$ have the same distribution, where $g(y) = |\det T|^{-1} f(T^{-1}y)$, $y \in TK$ and \hat{K}'_n is the convex hull of n i.i.d. points with density g . In addition, $d_f(K, \hat{K}_n) \geq \frac{m}{M} \frac{|K \setminus \hat{K}_n|}{|K|}$. Therefore, the same*

reasoning as in the proof of Theorem 4 yields

$$\mathbb{P}_f \left[n \left(\frac{|K \setminus \hat{K}_n|}{|K|} - \aleph'_1 n^{-2/(d+1)} \right) > x \right] \leq \aleph'_2 e^{-\aleph'_3 x}, \forall x > 0,$$

where \aleph'_1, \aleph'_2 and \aleph'_3 are positive constant that depend only on d and on the ratio M/m .

A drawback of Theorem 4 is that it involves constants which depend at least exponentially on the dimension d . However, this seems to be the price for getting a uniform deviation inequality on \mathcal{K}_d .

The following moment inequalities are a consequence of Theorem 4.

Corollary 3. *For every positive number q , there exists a positive constant A_q , which depends on d and q only, such that*

$$\sup_{K \in \mathcal{K}_d} \mathbb{E}_K \left[\left(\frac{|K \setminus \hat{K}_n|}{|K|} \right)^q \right] \leq A_q n^{-2q/(d+1)}. \quad (13)$$

Note that this corollary could also be derived from Vu's result [22], combined with Giannopoulos and Tsolomitis' result [8] (see Remark 3 below).

Combining Corollary 3 with the lower bound for the minimax risk obtained in [13] yields the following result.

Corollary 4. *For every positive number q , there exist positive constants a_q and A_q , which depend on d and q only, such that*

$$a_q n^{-2q/(d+1)} \leq \sup_{K \in \mathcal{K}_d} \mathbb{E}_K \left[\left(\frac{|K \setminus \hat{K}_n|}{|K|} \right)^q \right] \leq A_q n^{-2q/(d+1)}.$$

Remark 3. *An analogous result to Theorem 4 could be derived from Vu's result (2) combined with an elegant result proven by Giannopoulos and Tsolomitis [8], Theorem 3.6. Let $K \in \mathcal{K}_d^d$ have volume one and ϕ be a non decreasing function defined on the positive real line. Then, the expectation $\mathbb{E}_K[\phi(|\hat{K}_n|)]$ is minimized when K is an ellipsoid. The key argument is that this expectation does not increase when K is replaced by its Steiner symmetral with respect to a hyperplane, and performing such a transform iteratively on K leads to a Euclidean ball at the limit. When ϕ is the indicator function of the interval (x, ∞) , for $x > 0$, Giannopoulos and Tsolomitis' results implies that $\mathbb{P}[V_n \geq x]$ is maximized when K is an ellipsoid. Hence, applying (2) to an ellipsoid of volume one yields to a uniform deviation inequality as in*

Theorem 4. However, the range for x would be much smaller than ours, which allows to capture the whole right tail of V_n . Yet, it would still yield similar bounds for the moments of V_n , as in Corollary 3.

Let $K \in \mathcal{K}_d$ and n, q be positive integers. Hölder inequality yields $\mathbb{E}_K [R_n^q] \geq \mathbb{E}_K [R_n]^q$ and, by Efron's identity (8),

$$\mathbb{E}_K [R_n^q] \geq n^q \mathbb{E}_K \left[\frac{|K \setminus \hat{K}_{n-1}|}{|K|} \right]^q.$$

Hence, one gets the following theorem:

Theorem 5. *Let n and q be positive integers. Then, for some positive constants b_q and B_q which depend on d and q only,*

$$b_q n^{\frac{q(d-1)}{d+1}} \leq \sup_{K \in \mathcal{K}_d} \mathbb{E}_K [R_n^q] \leq B_q n^{\frac{q(d-1)}{d+1}}.$$

Combined with Theorem 3, this theorem has two consequences. First, the rate in the upper bound of Theorem 3 is tight, uniformly on all arbitrary convex bodies and bounded densities. Second, the uniform case is the worst case, i.e., yields the largest possible rate for the expected number of vertices of \hat{K}_n . Namely, the following holds. For $K \in \mathcal{K}_d^1$ and $M > 0$, denote by $\mathcal{F}(K, M)$ the collection of all densities that are supported on K and bounded by M . For two positive sequences u_n and v_n , write $u_n = O(v_n)$ if the ratio u_n/v_n is bounded, independently of n .

Theorem 6. *For all $M > 0$ and all positive integer q ,*

$$\sup_{K \in \mathcal{K}_d^1} \sup_{f \in \mathcal{F}(K, M)} \mathbb{E}_f [R_n^q] = O \left(\sup_{K \in \mathcal{K}_d^1} \mathbb{E}_K [R_n^q] \right).$$

Remark 4. *We do not know whether the supremum over K could be removed in Theorem 6: We propose the following open question. Is it true that for all $M > 0$ and $K \in \mathcal{K}_d^1$,*

$$\sup_{f \in \mathcal{F}(K, M)} \mathbb{E}_f [R_n] = O(\mathbb{E}_K [R_n]) \quad ?$$

6 Appendix: Proof of the lemmas

Proof of Lemma 2: Let $G \in \mathcal{K}_d$. Steiner formula (see Section 4.1 in [18]) states that there exist positive numbers $L_1(G), \dots, L_d(G)$, such that

$$|G^\lambda \setminus G| = \sum_{j=1}^d L_j(G) \lambda^j, \lambda \geq 0. \quad (14)$$

Besides the $L_j(G), j = 1, \dots, d$ are increasing functions of G . In particular, if $G \in \mathcal{K}_d^1$, then $L_j(G) \leq L_j(B_d)$.

Let $G, G' \in \mathcal{K}_d^1$, and let $\lambda = d_H(G, G')$. Since G and G' are included in the unit ball, λ is not greater than its diameter, so $\lambda \leq 2$. By definition of the Hausdorff distance, $G \subseteq G'^\lambda$ and $G' \subseteq G^\lambda$. Hence,

$$\begin{aligned} |G \triangle G'| &= |G \setminus G'| + |G' \setminus G| \leq |G'^\lambda \setminus G'| + |G^\lambda \setminus G| \\ &\leq 2 \sum_{j=1}^d L_j(B_d) \lambda^j \leq \lambda \sum_{j=1}^d L_j(B_d) 2^j. \end{aligned}$$

The Lemma is proved by setting $\alpha_1 = \sum_{j=1}^d L_j(B_d) 2^j$.

Note that since $\delta \leq 1$, Steiner formula (14) implies, for $G \in \mathcal{K}_d^1$, that

$$|G^\delta \setminus G| \leq \alpha_2 \delta, \quad (15)$$

where $\alpha_2 = \sum_{j=1}^d L_j(B_d)$. \square

Proof of Lemma 3: For $x \in K$,

$$f(x) = \frac{\text{Vol}_{D-d}((x+H) \cap K_0)}{\text{Vol}_D(K_0)},$$

where, for all integers p , Vol_p stands for the p -dimensional volume and we recall that H is the orthogonal space of \mathbb{R}^d in \mathbb{R}^D . Let $x \in K$ with $t = \rho(x, \partial K) \leq r$. Let $x' \in \partial K$ such that $\rho(x, \partial K) = \rho(x, x')$. Let $x_0 \in \partial K_0$ whose orthogonal projection onto \mathbb{R}^d is x' . By the r -rolling condition, there exists $a \in K_0$ with $x' \in B_D(a, r) \subseteq K_0$. Note that $x' - a \in \mathbb{R}^d$ (\mathbb{R}^d being identified to a subspace of \mathbb{R}^D , orthogonal to H) since the (unique) tangent space to K_0 at x' needs to be tangent to $B_D(a, r)$ as well. Therefore,

$$\text{Vol}_{D-d}((x+H) \cap K_0) \geq \text{Vol}_{D-d}((x+H) \cap B_D(a, r))$$

and $(x+H) \cap B_D(a, r)$ is a $(D-d)$ -dimensional ball with radius h , where $h = \sqrt{2rt - t^2} \geq \sqrt{rt}$. Hence, for all $x \in K$ with $\rho(x, \partial K) \leq r$,

$$f(x) \geq \frac{(rt)^{(D-d)/2} \beta_{D-d}}{\text{Vol}_D(K_0)} \geq (rt)^{(D-d)/2} \beta_{D-d} \beta_D,$$

which proves the lemma. \square

Proof of Lemma 4: For all $t \in (0, t_0]$,

$$\begin{aligned}
|K \setminus \hat{K}_n| &= \int_K \mathbb{1}_{x \notin \hat{K}_n} dx \\
&= \int_{K_f(t)} \mathbb{1}_{x \notin \hat{K}_n} dx + \int_{K \setminus K_f(t)} \mathbb{1}_{x \notin \hat{K}_n} dx \\
&\leq |K_f(t)| + \frac{1}{t} \int_{K \setminus K_f(t)} (x) \mathbb{1}_{x \notin \hat{K}_n} dx \\
&\leq Lt^\alpha + \frac{1}{t} d_f(K, \hat{K}_n).
\end{aligned} \tag{16}$$

If $d_f(K, \hat{K}_n) \leq t_0^{\alpha+1}$, take $t = d_f(K, \hat{K}_n)^{\alpha+1}$ in (16). \square

Proof of Lemma 5: For precision's sake, we denote by $\mathbb{P}^{\otimes n}$ the n -product of the probability measure μ , i.e., the joint probability measure of the random variables X_1, \dots, X_n , and by $\mathbb{E}^{\otimes n}$ the corresponding expectation operator. First, note that the expectation $\mathbb{E}^{\otimes n}[(1 - \mu(\hat{K}_n))^q]$ can be rewritten as :

$$\begin{aligned}
\mathbb{E}^{\otimes n} \left[(1 - \mu(\hat{K}_n))^q \right] &= \mathbb{E}^{\otimes n} \left[\mathbb{P}^{\otimes q} \left[X_{n+1} \notin \hat{K}_n, \dots, X_{n+q} \notin \hat{K}_n \mid X_1, \dots, X_n \right] \right] \\
&= \mathbb{P}^{\otimes(n+q)} \left[X_{n+j} \notin \hat{K}_n, \forall j = 1, \dots, q \right].
\end{aligned} \tag{17}$$

Using the symmetric role of X_1, \dots, X_{n+q} , and since the event $\{X_{n+j} \notin \hat{K}_n, \forall j = 1, \dots, q\}$ contains the event $\{X_{n+j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q\}$, (17)

yields

$$\begin{aligned}
& \mathbb{E}^{\otimes n} \left[(1 - \mu(\hat{K}_n))^q \right] \\
& \geq \mathbb{P}^{\otimes n+q} [X_{n+j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q] \\
& = \frac{1}{\binom{n+q}{q}} \sum_{1 \leq i_1 < \dots < i_q \leq n+q} \mathbb{P}^{\otimes n+q} [X_{i_j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q] \\
& = \frac{1}{\binom{n+q}{q}} \mathbb{E}^{\otimes n+q} \left[\sum_{1 \leq i_1 < \dots < i_q \leq n+q} \mathbf{1} (X_{i_j} \in \mathcal{V}_{n+q}, \forall j = 1, \dots, q) \right] \\
& = \frac{1}{\binom{n+q}{q}} \mathbb{E}^{\otimes n+q} \left[\binom{R_{n+q}}{q} \right] \\
& = \frac{\mathbb{E}^{\otimes n+q} [R_{n+q}(R_{n+q} - 1) \dots (R_{n+q} - q + 1)]}{(n+q)(n+q-1) \dots (n+1)},
\end{aligned}$$

which proves the lemma. \square

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